

Finite Math - Fall 2018

Lecture Notes - 11/6/2018

HOMEWORK

- Section 4.4 - 55, 57, 58
- Section 4.5 - 19, 20, 25, 26, 29, 32, 33, 35, 39, 42, 45, 51, 55, 61, 64

SECTION 4.4 - MATRICES: BASIC OPERATIONS

Example 1. Find a , b , c , and d such that

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

Solution. If we multiply out the matrices on the left, we get the equation

$$\begin{bmatrix} 6a - 5c & 6b - 5d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

And so we have the following two systems of equations

$$\begin{aligned} 6a - 5c &= -16 \\ 3c &= 24 \end{aligned}$$

and

$$\begin{aligned} 6b - 5d &= 64 \\ 3d &= -6 \end{aligned}$$

The augmented matrices for these two systems are

$$\left[\begin{array}{cc|c} 6 & -5 & -16 \\ 0 & 3 & 24 \end{array} \right] \quad \text{and} \quad \left[\begin{array}{cc|c} 6 & -5 & 64 \\ 0 & 3 & -6 \end{array} \right]$$

Notice that these two systems have the same coefficient matrix! In the first augmented matrix, we are aiming to solve for the variables a and c which are the first column of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and in the second augmented matrix, we are solving for the variables b and d . Because of this, we can just stick the augmented matrices together to form the new one

$$\left[\begin{array}{cc|cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array} \right]$$

Then, as before, we aim to get a reduced form on the left side, and that will simultaneously solve both systems. That is, we are aiming for

$$\left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array} \right]$$

So, let's solve the system

$$\begin{aligned} \left[\begin{array}{cc|cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array} \right] &\xrightarrow[\sim]{\frac{1}{6}R_1 \rightarrow R_1} \left[\begin{array}{cc|cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 3 & 24 & -6 \end{array} \right] &\xrightarrow[\sim]{\frac{1}{3}R_2 \rightarrow R_2} \left[\begin{array}{cc|cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 1 & 8 & -2 \end{array} \right] \\ &\xrightarrow[\sim]{R_1 + \frac{5}{6}R_2 \rightarrow R_1} \left[\begin{array}{cc|cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 8 & -2 \end{array} \right] \end{aligned}$$

So, we get that $a = 4$, $b = 9$, $c = 8$, and $d = -2$.

Example 2. Find a , b , c , and d such that

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{5}{18} \\ 0 & \frac{1}{3} \end{bmatrix}$$

SECTION 4.5 - INVERSE OF A SQUARE MATRIX

The Identity Matrix.

Example 3. Find the products

$$(a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.

(a)

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 1+0 & 2+0 \\ 0+3 & 0+4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1+0 & 0+2 \\ 3+0 & 0+4 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2+0 & 4+0 & 6+0 \\ 0+3 & 0+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0+0 & 0+4+0 & 0+0+6 \\ 3+0+0 & 0+6+0 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Definition 1 (Identity Matrix). *The $n \times n$ identity matrix is a matrix, denoted by I or I_n , which is an $n \times n$ matrix with 1's on the principal diagonal and 0's everywhere else. For an $m \times n$ matrix M , we have*

$$I_m M = M = M I_n.$$

From the above example, we have the two forms of the identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of a Square Matrix. It is only possible to find a multiplicative inverse of a matrix if it is a square matrix. So, we now restrict ourselves to square matrices.

Definition 2 (Inverse Matrix). *If M is a square matrix of size n , and if there is a matrix, denoted M^{-1} , such that*

$$M M^{-1} = M^{-1} M = I_n$$

we call M^{-1} the inverse of M . If M does not have an inverse, then M is called a singular matrix.

Example 4. *Find the inverse of the matrix*

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Solution. *Since we're looking for a matrix which satisfies $MX = I_n$, we can use the trick from the previous section and just row reduce $[M|A]$ to the form $[I_n|X]$, then check that $XM = I_n$ as well.*

$$\begin{aligned} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] &\xrightarrow[R_1 \leftrightarrow R_2]{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array} \right] &\xrightarrow[R_2 - 2R_1 \rightarrow R_2]{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array} \right] \\ &\xrightarrow[-R_2 \rightarrow R_2]{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array} \right] &\xrightarrow[R_1 - 2R_2 \rightarrow R_1]{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array} \right] \end{aligned}$$

So we should have that $M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. We need to check that $M^{-1}M = I_2$.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 6-6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Thus we have that

$$M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Example 5. Find the inverse of the matrix

$$N = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array} \right] \xrightarrow[R_2-2R_1 \rightarrow R_2]{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array} \right]$$

Since there is only zeros in the bottom line of the left side, the matrix N does not have an inverse.

Example 6. Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & -3 \\ -2 & -3 & -1 \end{bmatrix}.$$

Solution.

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 1 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[R_1 \leftrightarrow R_2]{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow[R_3+R_2 \rightarrow R_3]{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array} \right] \xrightarrow[R_2-2R_1 \rightarrow R_2]{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -2 & 6 & 1 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array} \right] \\ & \xrightarrow[R_2-3R_3 \rightarrow R_2]{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array} \right] \xrightarrow[R_3+R_2 \rightarrow R_3]{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array} \right] \\ & \xrightarrow[R_1-2R_2 \rightarrow R_1]{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array} \right] \xrightarrow[\frac{1}{8}R_3 \rightarrow R_3]{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right] \\ & \xrightarrow[R_1+21R_3 \rightarrow R_1]{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right] \xrightarrow[R_1+21R_3 \rightarrow R_1]{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 1 & -\frac{7}{8} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right] \end{aligned}$$

Theorem 1. *To find the inverse of an $n \times n$ matrix M , one begins with the augmented matrix $[M|I_n]$ and uses row operations to transform it into $[I_n|M^{-1}]$. However, if one or more rows of all 0's appear on the left side of the augmented matrix, M is not invertible, i.e., M^{-1} does not exist.*

Example 7. *Find the inverse of the following matrices (if possible):*

(a)

$$M = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

(b)

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

(c)

$$N = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

Remark 1. *There is a trick to invert a 2×2 matrix. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then*

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$